J. DIFFERENTIAL GEOMETRY
40 (1994) 1-6

REMARKS ON COMPLETE DEFORMABLE HYPERSURFACES IN R^4

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Dedicated to Professor T. Otsuki on his 75th birthday and to Professor S. Ishihara on his 70th birthday

Abstract

It is shown that, for each pair $\{k_1(u), k_2(v)\}$ of smooth functions on R with some conditions, there exists a family of complete nonruled deformable hypersurfaces $M(\lambda, k_1, k_2)$, $-\frac{1}{2} < \lambda < \frac{1}{2}$, in Euclidean space R^4 with rank $\rho = 2$ almost everywhere. This is an answer to one of the problems in [3].

1. Introduction and statement of results

It is an interesting problem to determine the deformability of an isometric immersion f of a connected Riemannian manifold M^n into Euclidean $(n+1)$-space R^{n+1}, $n \geq 3$. Let ρ be the rank of the second fundamental form of f. It is known (see [2]) that f is rigid (i.e., not deformable) if $\rho \geq 3$ by the Beez–Killing Theorem, and highly deformable if $\rho \leq 1$. The situation for constant rank $\rho = 2$ is quite complicated. Sbrana and Cartan divided this situation into three different types, and looked into it by a detailed local analysis (see [1], [4]).

It has been shown by Dajczer and Gromoll [3] that for $n \geq 3$ a complete hypersurface M^n in R^{n+1} whose set of all the geodesic points does not disconnect M^n, is rigid unless it contains either an open subset $L^3 \times R^{n-3}$ with L^3 unbounded or a complete ruled strip. But the three-dimensional case of this result remains an open problem.

In this paper, we construct a one-parameter family of complete nonruled deformable hypersurfaces in R^4 with rank $\rho = 2$ almost everywhere depending on two functions on the real line R with some conditions.

Theorem. *Let $k_j(x)$, $j = 1, 2$, be smooth functions on R satisfying that $-\frac{\pi}{4} < \int_0^x k_j(x)\,dx < \frac{\pi}{4}$, $j = 1, 2$, $\forall x \in R$ and that $k_1(u) > 0$, $k_2(v) < 0$ at all points u, v except for isolated ones. For each constant*

Received November 3, 1992.

λ, $-\frac{1}{2} < \lambda < \frac{1}{2}$, there exists an immersion $f(\lambda, k_1, k_2)$ of R^3 into R^4 satisfying the following conditions:

1. *The induced metric $ds^2(\lambda, k_1, k_2)$ on R^3 through $f(\lambda, k_1, k_2)$ is complete.*
2. *For any two constants λ, μ in $(-\frac{1}{2}, \frac{1}{2})$ the Riemannian manifolds $(R^3, ds^2(\lambda, k_1, k_2))$ and $(R^3, ds^2(\mu, k_1, k_2))$ are isometric.*
3. *For any two pairs of functions $\{k_1(x), k_2(x)\}$ and $\{\overline{k}_1(x), \overline{k}_2(x)\}$ and for two constants λ, μ in $(-\frac{1}{2}, \frac{1}{2})$ the isometric immersions $f(\lambda, k_1, k_2)$: $(R^3, ds^2(\lambda, k_1, k_2)) \to R^4$ and $f(\mu, \overline{k}_1, \overline{k}_2)$: $(R^3, ds^2(\mu, \overline{k}_1, \overline{k}_2)) \to R^4$ are congruent if and only if $\overline{k}_j(x) = k_j(\varepsilon_j x + a_j)$ for $\forall x \in R$, where $\varepsilon_j = \pm 1$ and a_j, $j = 1, 2$ are constants.*
4. *The rank $\rho(\lambda, k_1, k_2)$ of the second fundamental form of the immersion $f(\lambda, k_1, k_2)$ at each point (u, v, t) of R^3 is 2 (resp. ≤ 1) when $k_1(u)k_2(v) < 0$ (resp. $k_1(u)k_2(v) = 0$).*

We are in the C^∞ category and refer the readers to [2] for the terminology.

2. Preliminaries

First, we will recall some basic definitions. Let f: $M^n \to R^{n+1}$ be an isometric immersion of a connected n-dimensional Riemannian manifold M^n into the Euclidean space R^{n+1}. The isometric immersion f is said to be *rigid* if, for any other isometric immersion h: $M^n \to R^{n+1}$, there exists a motion τ of R^{n+1} such that $h = \tau \circ f$. The isometric immersion f: $M^n \to R^{n+1}$ which is not rigid is said to be *deformable*.

Let $k_j(x)$, $j = 1, 2$, be functions as in the Theorem. We define the two functions $\theta(u)$ and $\phi(u)$ by

$$\theta(u) = \int_0^u k_1(x)\,dx, \qquad \phi(v) = \int_0^v k_2(x)\,dx$$

for $u, v \in R$. For each constant λ in $(-\frac{1}{2}, \frac{1}{2})$ we define the functions $\theta(u, \lambda)$, $\phi(v, \lambda)$, $k_1(u, \lambda)$, $k_2(v, \lambda)$ by

$$\theta(u, \lambda) = \arcsin\left\{\sin\theta(u)/\sqrt{1-\lambda}\right\}, \qquad u \in R, \tag{2.1}$$

$$\phi(v, \lambda) = \arcsin\left\{\sin\phi(v)/\sqrt{1+\lambda}\right\}, \qquad v \in R, \tag{2.2}$$

$$(2.3)\qquad k_1(u,\lambda)=\frac{d}{du}\theta(u,\lambda),\qquad u\in R;$$

$$(2.4)\qquad k_2(u,\lambda)=\frac{d}{dv}\phi(v,\lambda),\qquad v\in R.$$

Denote by $c_1(u,\lambda)$, $e_1(u,\lambda)$, $e_2(u,\lambda)$ (resp. $c_2(v,\lambda)$, $e_3(v,\lambda)$, $e_4(v,\lambda)$) the curve in $R^2\times\{(0,0)\}$ (resp. $\{(0,0)\}\times R^2)\subset R^4$ and its Frenet frame with curvature $k_1(u,\lambda)$ (resp. $k_2(v,\lambda)$) and initial conditions:

$c_1(0,\lambda)=(0,\cdots,0)$, $e_1(0,\lambda)=(1,0,0,0)$, $e_2(0,\lambda)=(0,1,0,0)$, (resp. $c_2(0,\lambda)=(0,\cdots,0)$, $c_3(0,\lambda)=(0,0,1,0)$, $c_4(0,\lambda)=(0,0,0,1)$).

We define a mapping f_λ: $R^3\to R^4$ by

(2.5)

$$\begin{aligned}f_\lambda(u,v,t)&=c_1(u,\lambda)\\&\quad+t\sqrt{\frac{1-\lambda}{2}}\{\sin\theta(u,\lambda)e_1(u,\lambda)+\cos\theta(u,\lambda)e_2(u,\lambda)\},\\&\quad+c_2(v,\lambda)\\&\quad+t\sqrt{\frac{1+\lambda}{2}}\{\sin\phi(v,\lambda)e_3(v,\lambda)+\cos\phi(v,\lambda)e_4(v,\lambda)\},\end{aligned}$$

for $u,v,t\in R$. Using (2.1)–(2.4) we can show that

$$\frac{\partial}{\partial u}f_\lambda(u,v,t)=e_1(u,\lambda),\qquad \frac{\partial}{\partial v}f_\lambda(u,v,t)=e_3(v,\lambda),$$

$$\begin{aligned}\frac{\partial}{\partial t}f_\lambda(u,v,t)&=\sqrt{\frac{1-\lambda}{2}}\{\sin\theta(u,\lambda)e_1(u,\lambda)+\cos\theta(u,\lambda)e_2(u,\lambda)\}\\&\quad+\sqrt{\frac{1+\lambda}{2}}\{\sin\phi(v,\lambda)e_3(v,\lambda)+\cos\phi(v,\lambda)e_4(v,\lambda)\},\end{aligned}$$

and that

$$\begin{aligned}\xi_\lambda(u,v)&=\{\cos^2\theta(u)+\cos^2\phi(v))\}^{-1/2}\\&\quad\cdot\{\sqrt{1+\lambda}\cos\phi(v,\lambda)e_2(u,\lambda)-\sqrt{1-\lambda}\cos\theta(u,\lambda)e_4(v,\lambda)\}\end{aligned}$$

is a field of unit normals along f_λ. From this observation together with (2.1)–(2.4) it follows that

$$(2.6)\quad f_\lambda^*\,ds^2_{\mathrm{can}}=du^2+dv^2+\sqrt{2}\sin\theta(u)\,du\,dt+\sqrt{2}\sin\phi(v)\,dv\,dt+dt^2,$$

so that

(2.7)

$$\left\langle\frac{\partial^2 f_\lambda}{\partial u^2},\xi_\lambda\right\rangle=k_1(u)\cos\theta(u)\sqrt{\frac{\cos^2\phi(v)+\lambda}{(\cos^2\theta(u)-\lambda)(\cos^2\theta(u)+\cos^2\phi(v))}},$$

(2.8)

$$\left\langle \frac{\partial^2 f_\lambda}{\partial v^2}, \xi_\lambda \right\rangle = -k_2(v)\cos\phi(v)\sqrt{\frac{\cos^2\theta(u)-\lambda}{(\cos^2\phi(v)+\lambda(\cos^2\theta(u)+\cos^2\phi(v))}},$$

$$\left\langle \frac{\partial^2 f_\lambda}{\partial u\partial v}, \xi_\lambda \right\rangle = \left\langle \frac{\partial f_\lambda}{\partial u\partial t}, \xi_\lambda \right\rangle = \left\langle \frac{\partial^2 f_\lambda}{\partial v\partial t}, \xi_\lambda \right\rangle = \left\langle \frac{\partial^2 f_\lambda}{\partial t^2}, \xi_\lambda \right\rangle = 0. \tag{2.9}$$

3. Proof of Theorem

We will maintain the notation as in the previous section. We will prove the first assertion. First, we see that, for each constant λ, $-\frac{1}{2} < \lambda < \frac{1}{2}$ the mapping f_λ given by (2.5) is an immersion by virtue of (2.6) and

$$-\pi/4 < \theta(u), \phi(v) < \pi/4, \quad \forall u, v \in R. \tag{3.1}$$

Set $g = f_\lambda^* ds^2_{\text{can}}$, and denote by g_{ij} the components of g with respect to the global coordinates $x_1 := u$, $x_2 := v$ and $x_3 := t$ on R. Then the solutions of the equation in ρ: $\det(\rho\delta_{ij} - g_{ij}) = 0$ are $\rho = 1$, $1 \pm \{[\sin^2\theta(u) + \sin^2\phi(v)]/2\}^{1/2}$. Using (3.1) we have

$$a g_{\text{can}}(X, X) \leq g(X, X) \leq b g_{\text{can}}(X, X) \tag{3.2}$$

for all tangent vectors X in R^3, where g_{can} is the canonical Riemannian metric on R^3, and a and b are positive constants satisfying that $a^2 = 1 - 1/\sqrt{2}$, $b^2 = 1 + 1/\sqrt{2}$. Thus (3.2) implies that the first assertion is true.

The second assertion is valid because of (2.6).

The third assertion is proved as follows. Let $\overline{\theta}(u, \lambda)$, $\overline{\phi}(v, \lambda)$, $\overline{k}_1(u, \lambda)$, $\overline{k}_2(v, \lambda)$, $\overline{c}_1(u, \lambda)$, $\overline{e}_i(u, \lambda)$, $i = 1, 2$, $\overline{c}_2(v, \lambda)$, $\overline{e}_i(v, \lambda)$, $i = 3, 4$, and $\overline{f}_\lambda$ be the corresponding functions, curves, Frenet frames and the mappings as in the previous section for $\overline{k}_1(u)$, $\overline{k}_2(v)$, and μ.

Suppose that there exist a diffeomorphism ψ of R^3 onto itself and an isometry ρ of (R^4, ds^2_{can}) such that

$$\rho \circ f_\lambda(u, v, t) = \overline{f}_\mu \circ \psi(u, v, t). \tag{3.3}$$

We can show that, for each fixed λ, $-\frac{1}{2} < \lambda < \frac{1}{2}$, a curve $u = u(\sigma)$, $v = v(\sigma)$, $t = t(\sigma)$, $\sigma \in R$ defines a geodesic in (R^3, g) and (R^4, ds^2_{can}) if and only if $u(\sigma) = \text{const}$, $v(\sigma) = \text{const}$, and $t(\sigma) = \pm\sigma + \text{const}$, provided that $k_1(u(\sigma_0))k_2(v(\sigma_0)) < 0$ for some σ_0. Notice that, for each fixed $u, v \in R$, the mapping $t \in R \to f_\lambda(u, v, t)$ (resp. $\overline{f}_\mu(u, v, t)$) defines a geodesic in (R^4, ds^2_{can}), and that for almost all (u, v) in R^2, $\overline{k}_1(\psi_1(u, v, 0))\overline{k}_2(\psi_2(u, v, 0)) < 0$, where $\psi_j(u, v, 0)$ is the jth component of $\psi(u, v, 0) \in R^3$.

From these observations, we may assume, by adding constants to the parameters and rotating $f_\lambda(R^3)$ around the origin if necessary, that

$$\begin{aligned} &\rho = \text{identity}, \\ &\psi(0, 0, 0) = (0, 0, 0), \quad \psi_u(0, 0, 0) = (1, 0, 0), \\ &\psi_v(0, 0, 0) = (0, 1, 0), \quad \psi_t(u, v, t) = (0, 0, 1), \end{aligned} \tag{3.4}$$

$\forall u, v, t \in R$, where ψ_u, ψ_v, and ψ_t are the partial derivatives of ψ with respect to u, v, and t respectively. From this we find that

$$\psi(u, v, t) = (x(u, v), y(u, v), t) \quad \forall u, v, t \in R, \tag{3.5}$$

where $x(u, v)$, $y(u, v)$ are functions of u and v.

On the other hand, for each fixed $t \in R$, the mapping $\imath(t)$: $R^2 \to R^3$, $(u, v) \mapsto (u, v, t)$ is an isometric imbedding of (R^2, g_{can}) into $(R^3, f_\lambda^* ds^2_{\text{can}})$, where g_{can} is the Euclidean metric on R^2. Combining this fact with (2.5), (3.5) shows that the mapping $(u, v) \mapsto (x(u, v), y(u, v))$ is an isometry of (R^2, g_{can}). Thus by this remark and (3.4),

$$\psi(u, v, t) = (u, v, t) \quad \forall u, v, t \in R. \tag{3.6}$$

From (3.3), (3.4), and (3.6) it follows that

$$\begin{cases} \overline{k}_i(x) = k_i(\varepsilon_i x + a_i), & \varepsilon_i, a_i\text{: constants, with } \varepsilon_i = \pm 1, \\ \mu = \lambda \end{cases} \tag{3.7}$$

for each $x \in R$.

Conversely, it can be easily shown that if (3.7) is satisfied, then we have (3.3) for some diffeomorphism $(u, v, t) \mapsto \psi(u, v, t)$. This completes the proof of the third assertion.

The fourth assertion follows easily from (2.7)–(2.9).

Acknowledgment

The author would like to thank Professor M. Dajczer for useful conversations.

References

[1] É. Cartan, *La déformation des hypersurfaces dans l'espace euclidean réel à n dimensions*, Bull. Soc. Math. France **44** (1916) 65–99.

[2] M. Dajczer et al., *Submanifolds and isometric immersions*, Publish or Perish, Houston, TX, 1990.

[3] M. Dajczer & D. Gromoll, *Rigidity of complete euclidean hypersurfaces*, J. Differential Geometry **31** (1990) 404–416.

[4] V. Sbrana, *Sulle varietá ad* $n-1$ *dimensioni deformabili nello spazio ad n dimensioni*, Rend. Circ. Mat. Palermo **27** (1909) 1–45.

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